Higher Traces, Hochschild Homology and Grothendieck-Riemann-Roch

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1 Introduction

For a finite free module M over a ring R, we have a dual module $M^\vee = \hom(M, R)$. If we choose a finite basis (r_i) for M, then we get a dual basis $(r_i^\vee)$ for $M^\vee$. We can then calculate the trace of an endomorphism $f \in \mathrm{End}_R(M)$ by the formula

$$\mathrm{tr}(f) = \sum_{i=1}^{n} \mathrm{ev}(e_i^\vee, f(e_i)) \in R.$$

which is an invariant of f. For example the dimension of M is equal to $\mathrm{tr}(\mathrm{id}_M)$.

We can generalize the notion of dualizable objects and traces to symmetric monoidal (∞, n)-categories, where the trace of an endomorphism will be an endomorphism of the unit. We are particularly interested in a categorified version of the first example, where we replace Ab by $\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}}$ the $(\infty, 2)$-category of presentable ∞-categories and colimit-preserving functors.

In this case, we have $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})$ and the symmetric monoidal $(\infty, 2)$-category $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})$. In this setting, we can identify for an $R \in \mathrm{Alg}(\mathcal{C})$, the (Topological) Hochschild homology $\mathrm{HH}(R) = R \otimes_{R \otimes R^{\mathrm{op}}} R \in \mathcal{C}$ with the trace of the dualizable object $\mathrm{RMod}_R(\mathcal{C}) \in \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})$.

The typical example of this is $\mathcal{C} = \mathrm{Sp}$, but we are interested in the example of $\mathcal{C} = \mathcal{D}\left(\mathrm{Liq}\right)_p$, the derived ∞-category of complex liquid vector spaces [CS22, Definition 2.13, Theorem 3.11].

For a compact complex manifold X, $\mathcal{O}_X$ which is the derived category of quasicoherent sheaves on X, is an algebra in $\mathrm{D}(\mathrm{Liq}_{\mathrm{p}})$, so we can take its Hochschild homology. We show that Hochschild homology, and that the trace is a localizing invariant.

This gives a map from the K-theory to Hochschild homology, and this is the first step to proving Grothendieck-Riemann-Roch.

2 Higher Traces

We will follow the exposition given in [CCRY22].

2.1 Definition of dualizable objects and traces

Definition 2.1. *Let* $(\mathcal{C}^{\otimes}$ *be a symmetric monoidal* ∞*-category. An object* X *is dualizable if there exist another object* $X^\vee$ *with maps* $\mathrm{ev}_X : X^\vee \otimes X \to \mathbb{1}$ *and* $\mathrm{coev}_X : \mathbb{1} \to X \otimes X^\vee$ *such that the composite maps*

$$X \xrightarrow{\mathrm{coev}_X \otimes id_X} X \otimes X^\vee \otimes X \xrightarrow{id_X \otimes \mathrm{ev}_X} X$$

$$X^\vee \xrightarrow{Id_X \otimes \mathrm{coev}_X} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{ev}_X \otimes Id_X} X$$

are homotopic to the identity.

Informally, dualizability is a finite condition on objects of an symmetric monoidal ∞-category.

Given a map $f : X \to Y$ between dualizable objects, we have a dual morphism given by the composite map

$$Y^\vee \xrightarrow{\mathrm{id} \otimes \mathrm{coev}_X} Y^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{id} \otimes f \otimes \mathrm{id}} Y^\vee \otimes Y \otimes X^\vee \xrightarrow{\mathrm{ev}_Y \otimes \mathrm{id}} X^\vee.$$

For an endomorphism of an dualizable object, we can define its trace.

Definition 2.2. *Let $f : X \to X$ be an endomorphism of a dualizable object. Then the trace $tr(f) \in \mathrm{Map}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is given by the composite map*

$$\mathbb{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^\vee \xrightarrow{f \otimes \mathrm{id}} X \otimes X^\vee \xrightarrow{\simeq} X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} \mathbb{1}$$

The dimension of X is the trace of the identity map.

Example 2.3. *Given a ring R, the dualizable objects of the derived ∞-category $\mathcal{D}(R)$ are the perfect complexes. The dimension of a perfect complex C is the Euler characteristic and the trace of an endomorphism is the Lefschetz number.*

In particular a finite free module $R^{\oplus n}$ is dualizable and the trace of an endomorphism of a finite free module is the trace from linear algebra.

similar to the trace in linear algebra, the trace is cyclicly invariant in general. Indeed, given morphisms $f : Y \to X$ and $g : X \to Y$ of dualizable objects, we have $\mathrm{tr}(fg) \simeq \mathrm{tr}(gf)$, defined by the following diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\mathrm{coev}_X"] \arrow[d, equal] & X \otimes X^\vee \arrow[r, "fg \otimes id_{X^\vee}"] \arrow[dr, "g \otimes f^\vee"] & X \otimes X^\vee \arrow[r, "\mathrm{ev}_X"] & \mathbb{1} \arrow[d, equal] \\
\mathbb{1} \arrow[r, "\mathrm{coev}_Y"'] & Y \otimes Y^\vee \arrow[r, "gf \otimes id_{X^\vee}"'] & Y \otimes Y^\vee \arrow[r, "\mathrm{ev}_Y"'] & \mathbb{1}.
\end{tikzcd}$$

We would like to make the trace into a natural transformation. In particular, we would like for a symmetric monoidal category $\mathcal{C}$, an anima $\mathcal{C}^{\mathrm{trl}}$ of endomorphisms of dualizable objects, with the trace a functor of anima

$$\mathcal{C}^{\mathrm{trl}} \to \Omega\mathcal{C}$$

that is natural in $\mathcal{C}$.

If $\mathcal{C}$ is a $(\infty, 1)$-category, then the endomorphisms of dualizable objects assemble to an anima and not an $(\infty, 1)$-category.

However if we start with an $(\infty, 2)$-category $\mathcal{C}$ (and more generally (∞, n)-categories), then the trace functor will be a functor of $(\infty, n-1)$-categories. We will first define the functor for the $(\infty, 1)$-case, and use it to construct the general (∞, n)-case.

2.2 $(\infty, 1)$-functoriality of the trace

Let $\Omega\mathcal{C} = \mathrm{Map}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ be the space of endomorphisms of the unit $\mathbb{1}$, and let $\mathcal{C}^{\mathrm{trl}} = \mathrm{Map}(B\mathbb{N}, \mathcal{C}^{\mathrm{dbl}})$ be the ∞-groupoid, where $\mathcal{C}^{\mathrm{dbl}}$ is the full subcategory spanned by dualizable objects. We want to construct a map of spaces

$$tr : \mathcal{C}^{\mathrm{trl}} \to \Omega\mathcal{C}$$

Which is natural in $\mathcal{C}$.

The key to defining this, is to show that the functor $(-)^{\mathrm{dbl}}$ is corepresentable, so that the we can construct the natural map by using the Yoneda lemma.

First $(-)^{\mathrm{dbl}}$ is a functor $\mathrm{CAlg}(\mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$, since monoidal functors preserve dualizable objects. This functor has a left adjoint[1], which we denote by $\mathrm{Fr}^{\mathrm{dbl}}(-)$. Thus, have an equivalence

$$\mathrm{Map}^{\otimes}(\mathrm{Fr}^{\mathrm{dbl}}(I), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Map}(I, \mathcal{C}^{\mathrm{dbl}}).$$

It follows that $(-)^{\mathrm{trl}}$ is corepresented by $\mathrm{Fr}^{\mathrm{dbl}}(B\mathbb{N})$, so by the Yoneda lemma the trace map is determined by a point of $\Omega\,\mathrm{Fr}^{\mathrm{dbl}}(B\mathbb{N})$.

The unit of the adjunction is a map $\gamma_{univ} : B\mathbb{N} \to \mathrm{Fr}^{\mathrm{dbl}}(B\mathbb{N})^{dbl}$, which, by definition, is an element of $\mathrm{Fr}^{\mathrm{dbl}}(B\mathbb{N})^{\mathrm{trl}}$. We call this endomorphism the universal traceable endomorphism. Taking the trace of this endomorphism gives an element of $\Omega\,\mathrm{Fr}^{\mathrm{dbl}}(B\mathbb{N})$, which by the Yoneda Lemma gives a natural transformation. We define this to be the trace map. The trace is now given by the following diagram.

$$\begin{tikzcd}
\mathrm{Fr}^{\mathrm{dbl}}(B\mathbb{N})^{\mathrm{trl}} \arrow[rrr, "\mathrm{tr}_{\mathrm{Fr}^{\mathrm{dbl}}(B\mathbb{N})^{\mathrm{trl}}}"] \arrow[ddd, "F\circ -"'] & & & \Omega\,\mathrm{Fr}^{\mathrm{dbl}}(B\mathbb{N}) \arrow[ddd, "\Omega F"] \\
& \gamma_{univ} \arrow[r] \arrow[d] & \mathrm{tr}(\gamma_{univ}) \arrow[d] & \\
& f \arrow[r] & \Omega F(\mathrm{tr}(\gamma_{univ})) & \\
\mathcal{C}^{\mathrm{trl}} \arrow[rrr, "\mathrm{tr}_{\mathcal{C}}"'] & & & \Omega\mathcal{C}
\end{tikzcd}$$

Since F is monoidal, we have

$$\Omega F(\mathrm{tr}(\gamma_{univ})) \simeq \mathrm{tr}(F \circ \gamma_{univ}) \simeq \mathrm{tr}(f),$$

showing that the natural map agrees with our former definition on objects. By precomposing by the map $(\mathcal{C}^{\mathrm{dbl}})^{\simeq} \hookrightarrow \mathcal{C}^{\mathrm{trl}}$ that to a dualizable object assigns its identity map, we also get a functorial assignment of dimension.

2.3 Quick facts about (∞, n)-categories

We will now define the trace functor for an (∞, n)-category $\mathcal{C}$.
In our context define (∞, n)-categories inductively as ∞-categories enriched in $(\infty, n-1)$-categories.

We will give some properties of (∞, n)-categories, which differ from ordinary $(\infty, 1)$-categories.

[1] Maxime showed this in the exercise session.

In $(\infty, 1)$-categories, k-morphisms are $(k-1)$-morphisms in the mapping space, and since all the maps in an anima are equivalences, all k-morphisms for $k \geq 1$ are equivalences. However this is not true in an (∞, n)-category, as a k-morphism is instead a $k-1$-morphism in the mapping $(\infty, n-1)$-category. Continuing in this way the k-morphism is an 1-morphism of an $(\infty, n-k-1)$-category for $k \leq n$, where it need not be an equivalence.

The primary example of an (∞, n)-category is the (∞, n)-category of $(\infty, n-1)$-categories with the mapping $(\infty, n-1)$-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_{(\infty, n-1)}$.

Based on this example, we define adjoint morphisms in an arbitrary (∞, n)-category.

Definition 2.4 (Adjoint maps)**.** *A map $L \in \mathrm{Map}_{\mathcal{C}}(X, Y)$ is left adjoint to a map $R \in \mathrm{Map}_{\mathcal{C}}(Y, X)$ for $\mathcal{C} \in \mathrm{Cat}_{(\infty,n)}$, if there exist 2-maps*

$$LR \xrightarrow{\epsilon} \mathrm{id}_Y$$
$$\mathrm{id}_X \xrightarrow{\eta} RL$$

satisfying the triangle identities

$$\begin{tikzcd} L \arrow[r, "L\eta"] \arrow[dr, "id_L"'] & LRL \arrow[d, "\epsilon L"] \\ & L \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\eta R"] \arrow[dr, "\mathrm{id}_R"'] & RLR \arrow[d, "R\epsilon"] \\ & R \end{tikzcd}$$

Note that for an $(\infty, 1)$-category all the 2-morphisms are equivalences, so an adjoint pair in this case will just be a pair of inverse morphisms.

2.4 Higher traces

We will now extend the trace to (∞, n)-categories $\mathcal{C}$, where the trace map will be a natural map of $(\infty, n-1)$-categories

$$\mathrm{tr} : \mathcal{C}^{\mathrm{trl}} \to \Omega\mathcal{C}$$

In particular, there are non-equivalent morphisms in $\mathcal{C}^{\mathrm{trl}}$ between two endomorphisms. A general morphism between two endomorphisms has the following description.

Definition 2.5. *A map of traceable endomorphisms f and g, is a diagram*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X \arrow[r, "\varphi"'] \arrow[ur, Rightarrow, "\alpha"] & Y, \end{tikzcd}$$

where φ is a left adjoint and $\alpha : \varphi f \to g\varphi$ is a 2-morphism.

We can then define how the trace should act on these morphisms:

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{coev}_X"] \arrow[d, dashed, "\mathrm{coev}_Y"'] \arrow[dr, phantom, "="] & X \otimes X^\vee \arrow[r, "f \otimes id_{X^\vee}"] \arrow[d, "1 \otimes (\phi^r)^\vee"] \arrow[dr, phantom, "\swarrow"] & X \otimes X^\vee \arrow[d, dashed, "\phi \otimes 1"] \arrow[dr, bend left, "id_{X^\vee X^\vee}"] \arrow[r, phantom, "\swarrow"] & {} \\
Y \otimes Y^\vee \arrow[r, dashed, "\phi^r \otimes 1"] \arrow[dr, bend right, "id_{Y^\vee Y^\vee}"'] \arrow[r, phantom, "\swarrow", shift right=4] & X \otimes Y^\vee \arrow[d, dashed, "\phi \otimes 1"] & Y \otimes X^\vee \arrow[r, dashed, "\phi^r \otimes 1"] \arrow[d, dashed, "1 \otimes (\phi^r)^\vee"] \arrow[dr, phantom, "="] & X \otimes X^\vee \arrow[d, dashed, "\mathrm{ev}_X"] \\
& Y \otimes Y^\vee \arrow[r, dashed, "g \otimes 1"] & Y \otimes Y^\vee \arrow[r, "\mathrm{ev}_Y"] & 1.
\end{tikzcd}$$

The equality in the upper left and lower right corners follow from definition of dual morphisms.

The optimal construction of $(-)^{\mathrm{trl}}$, would be as a corepresentable functor, since we could then define the trace again by the Yoneda Lemma. However it turns out that corepresentable functors are to rigid, so we have to introduce the lax and oplax functor ∞-categories.

Recall that for a natural transformation η between two functors $F, G : \mathcal{C} \to \mathcal{D}$ there should be for every morphism $f : x \to y$ in $\mathcal{C}$ a commuting diagram

$$\begin{tikzcd}
F(x) \arrow[r, "\eta_x"] \arrow[d, "Ff"'] & G(x) \arrow[d, "Gf"] \\
F(x) \arrow[r, "\eta_y"'] \arrow[ur, Rightarrow, "\simeq"] & G(y),
\end{tikzcd}$$

where the commutativity is given by 2-equivalence. If we remove the criteria of this morphism being an equivalence, we get a lax commuting square, and if we instead had required the 2-morphism to go the other way, like so

$$\begin{tikzcd}
F(x) \arrow[r, "\eta_x"] \arrow[d, "Ff"'] & G(x) \arrow[d, "Gf"] \arrow[dl, Rightarrow] \\
F(x) \arrow[r, "\eta_y"'] & G(y),
\end{tikzcd}$$

we get an oplax commuting square. Replacing the conditions of coherences given by equivalences in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, with any morphisms then gives us two new categories $\mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}_{\mathrm{oplax}}(\mathcal{C}, \mathcal{D})$ depending on which way the arrows are oriented.

Notice that the conditions on functors have not changed, so the categories have the same objects as $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, but different higher morphisms.

Defining these categories formally is a bit technical, since we need to describe all the coherences. The ideal approach, which has not yet been fully developed, is to give a non-symmetric monoidal structure on $\mathrm{Cat}_{(\infty,n)}$ called the Gray product denoted by $\times^{\mathrm{lax}}$, such that we can define the categories by representability:

$$\mathrm{Map}(\mathcal{E}, \mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Map}(\mathcal{E} \times^{\mathrm{lax}} \mathcal{C}, \mathcal{D})$$
$$\mathrm{Map}(\mathcal{E}, \mathrm{Fun}_{\mathrm{oplax}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Map}(\mathcal{C} \times^{\mathrm{lax}} \mathcal{E}, \mathcal{D}).$$

So far the Gray product has only been defined for $(\infty,2)$-categories [GHL20] and a set $(\theta^{\vec{k}})$ generating $\mathrm{Cat}_{(\infty,n)}$ by colimits [JFS17, Definition 5.7]. However the special case $\theta^{\vec{k}}$ is enough to build the lax and oplax functor categories. Avoiding the technical details, we then have (∞,n)-categories

$$\mathrm{Fun}_{\mathrm{lax}}(\mathcal{C},\mathcal{D}),\mathrm{Fun}_{\mathrm{oplax}}(\mathcal{C},\mathcal{D})$$

with objects the same as for $\mathrm{Fun}(\mathcal{C},\mathcal{D})$, but with oplax and lax natural transformation.

Remark 2.6. *If $\mathcal{D}$ is a symmetric monoidal category, then $\mathrm{Fun}_{\mathrm{oplax}}(\mathcal{C},\mathcal{D})$ and $\mathrm{Fun}_{\mathrm{lax}}(\mathcal{C},\mathcal{D})$ inherit the symmetric monoidal structure of $\mathcal{D}$, since the functors $\mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}-)$ and $\mathrm{Fun}_{\mathrm{oplax}}(\mathcal{C},-)$ preserve limits and in particular finite products, and so preserve monoids in $\mathrm{Cat}_{(\infty,n)}$.*

We can also define the lax and oplax functor categories of monoidal structures on $\mathcal{C},\mathcal{D}$.

Definition 2.7. *For symmetric monoidal (∞,n)-categories $\mathcal{C}$ and $\mathcal{D}$, the (∞,n)-categories $\mathrm{Fun}^{\otimes}_{\mathrm{lax}}(\mathcal{C},\mathcal{D})$ $\mathrm{Fun}^{\otimes}_{\mathrm{lax}}(\mathcal{C},\mathcal{D})$ are given by*

$$\mathrm{Map}(\mathcal{E},\mathrm{Fun}^{\otimes}_{\mathrm{lax}}(\mathcal{C},\mathcal{D}))\simeq \mathrm{Map}^{\otimes}(\mathcal{C},\mathrm{Fun}_{\mathrm{oplax}}(\mathcal{E},\mathcal{D}))$$
$$\mathrm{Map}(\mathcal{E},\mathrm{Fun}^{\otimes}_{\mathrm{oplax}}(\mathcal{C},\mathcal{D}))\simeq \mathrm{Map}^{\otimes}(\mathcal{C},\mathrm{Fun}_{\mathrm{lax}}(\mathcal{E},\mathcal{D})).$$

Using these oplax transformations we can define an appropriate $(\infty,n-1)$-category $\mathcal{C}^{\mathrm{trl}}$.

Definition 2.8. *Given a symmetric monoidal (∞,n)-category $\mathcal{C}$, the $(\infty,n-1)$-category $\mathcal{C}^{\mathrm{trl}}$ is given by*

$$\mathcal{C}^{\mathrm{trl}} := \iota_{n-1}\,\mathrm{Fun}^{\otimes}_{\mathrm{oplax}}(\mathrm{Fr}^{\mathrm{dbl}}(B\mathbb{N}),\mathcal{C}),$$

where ι_m takes the maximal sub-(∞,m)-category of an (∞,n)-category.

We have a natural isomorphism

$$\iota_0\,\mathrm{Fun}^{\otimes}_{\mathrm{oplax}}(-,-)\simeq \mathrm{Map}^{\otimes}(-,-)$$

as functors landing in anima. Informally, this can seen since a natural isomorphism in $\mathrm{Fun}^{\otimes}_{\mathrm{oplax}}(-,-)$ will consist of equivalences, and as such correspond to a morphism of $\mathrm{Map}^{\otimes}(-,-)$.

As such, we get for an $(\infty,1)$-category $\mathcal{C}$, that our two definitions of $\mathcal{C}^{\mathrm{trl}}$ agree. The following corollary shows that this construction agrees with our notions of objects and morphisms from before.

Corollary 2.9. *There is a monomorphism of (∞,n)-categories*

$$\mathcal{C}^{\mathrm{trl}}\hookrightarrow \mathrm{Fun}_{\mathrm{oplax}}(B\mathbb{N},\mathcal{C}),$$

which on objects are the endomorphisms of dualizable objects, and morphisms are as in definition 2.5

This shows that we have an appropriate source $(\infty, n-1)$-category. We can also construct the $(\infty, n-1)$-category of dualizable objects

$$\mathcal{C}^{\mathrm{dbl}} := \iota_{n-1} \operatorname{Fun}^{\otimes}_{\mathrm{oplax}}(\mathrm{Fr}^{\mathrm{dbl}}(*), \mathcal{C}).$$

From the map $B\mathbb{N} \to *$, we get an embedding $\mathcal{C}^{\mathrm{dbl}} \to \mathcal{C}^{\mathrm{trl}}$

Definition 2.10 (Loop Space)**.** *The loop space $\Omega\mathcal{C}$ of an (∞, n)-category is defined by the following pullback in* $\mathrm{Cat}_{(\infty,n-1)}$

$$\begin{tikzcd}
\Omega\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}_1 \arrow[d, "{(d_1,d_0)}"] \\
\mathbb{1} \arrow[r, "{(\mathbb{1},\mathbb{1})}"] & \mathcal{C}_0 \times \mathcal{C}_0.
\end{tikzcd}$$

This matches our definition for $(\infty, 1)$-categories. The objects are the endomorphisms of the unit, and morphisms are 2-morphisms between 2 endomorphisms, etc. If $\mathcal{C}$ is symmetric monoidal, then $\Omega\mathcal{C}$ is a pullback of symmetric monoidal categories, and as such there is a symmetric monoidal structure on $\Omega\mathcal{C}$. We now have the source and target categories of the trace functor. To construct the trace functor, we reduce it to the $n = 1$ case by the two following lemmas.

Lemma 2.11. *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. For any $(\infty, n-1)$-category $\mathcal{E}$ there is an equivalence*

$$\mathrm{Map}(\mathcal{E}, \mathcal{C}^{\mathrm{trl}}) \simeq (i_1 \operatorname{Fun}_{\mathrm{lax}}(\mathcal{E}, \mathcal{C}))^{\mathrm{trl}}.$$

Lemma 2.12. *For every $(\infty, n-1)$-category $\mathcal{E}$ the canonical map*

$$\operatorname{Fun}_{\mathrm{lax}}(\mathcal{E}, \Omega\mathcal{C}) \to \Omega \operatorname{Fun}_{\mathrm{lax}}(\mathcal{E}, \mathcal{C})$$

is an equivalence. In particular,

$$\Omega(i_1 \operatorname{Fun}_{\mathrm{lax}}(\mathcal{E}, \mathcal{C})) \simeq \mathrm{Map}(\mathcal{E}, \Omega\mathcal{C}).$$

We can now define the trace functor

Definition 2.13. *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. The trace functor*

$$tr : \mathcal{C}^{\mathrm{trl}} \to \Omega\mathcal{C}$$

is the functor that represents the composite map

$$\mathrm{Map}(\mathcal{E}, \mathcal{C}^{\mathrm{trl}}) \simeq (i_1 \operatorname{Fun}_{\mathrm{lax}}(\mathcal{E}, \mathcal{C}))^{\mathrm{trl}} \xrightarrow{tr} \Omega(i_1 \operatorname{Fun}_{\mathrm{lax}}(\mathcal{E}, \mathcal{C})) \simeq \mathrm{Map}(\mathcal{E}, \Omega\mathcal{C})$$

with $\mathcal{E} \in Cat_{(\infty,n-1)}$:

Since the functor is defined by our original trace functor, it agrees with it on objects. It can also be shown that it agrees with the previous definition of trace of map of endomorphisms. Precomposing with the embedding $\mathcal{C}^{\mathrm{dbl}} \hookrightarrow \mathcal{C}^{\mathrm{trl}}$ we get the higher dimension map.

3 Traces of $\mathcal{C}$-linear ∞-categories

With the higher categorical trace, we can look at a categorified version of our original example, the module category $\mathrm{Mod}_R(Ab)$ for a ring R. In this categorified setting, we will show how to identify the Hochschild homology of a ring R with a trace.

Let $\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}}$ be the $(\infty, 2)$-category of presentable $\infty, 1$-categories, and let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})$ be a presentably stable symmetric monoidal ∞-category. We can form the $(\infty, 2)$-category $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})$ of presentable $\mathcal{C}$-linear categories. The morphisms of $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})$ are the $\mathcal{C}$-linear left adjoint functors, and the 2-morphisms are the $\mathcal{C}$-linear natural transformations.

This $(\infty, 2)$-category share a lot of the properties of ordinary module categories. For instance there is a $\mathcal{C}$-linear version of the Lurie tensor product $\mathcal{D} \otimes_{\mathcal{C}} \mathcal{E}$, which is the universal recipient of a $\mathcal{C}$-linear colimit-preserving map

$$\mathcal{D} \times \mathcal{E} \to \mathcal{D} \otimes_{\mathcal{C}} \mathcal{E},$$

So $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})$ is a symmetric monoidal category. Furthermore we have an action on $\mathrm{Fun}_{\mathcal{C}}(\mathcal{E}, \mathcal{D})$ by $\mathcal{C}$, so it is a $\mathcal{C}$-linear category, and therefore is an internal hom object in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})$.

Given a traceable endomorphism $\mathcal{E} \in \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})$, we can calculate its trace, which lands in $\Omega\, \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}}) \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}, \mathcal{C})$. However, as we have discussed earlier $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) \simeq \mathcal{C}$ by evaluating at the unit $\mathbb{1} \in \mathcal{C}$.

Taking the composition with this equivalence, we can write the trace functor for this category as:

Definition 3.1. *The $\mathcal{C}$-linear trace functor is the composite map*

$$\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})^{\mathrm{trl}} \xrightarrow{\mathrm{tr}_{\mathrm{Mod}_{\mathcal{C}}}} \Omega\mathcal{C} \xrightarrow{\simeq} \mathcal{C}.$$

Example 3.2. *As $\mathcal{C}$ is the unit of $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})$, it is a dualizable object. Considering the endomorphism given by tensoring with $X \in \mathcal{C}$, which we denote by X, and writing out the trace, we find that $\mathrm{Tr}_{\mathcal{C}} X \simeq X$.*

3.1 Hochschild homology as a $\mathcal{C}$-linear trace

With the proper framework done, we can now identify the Hochschild homology with the trace of a certain object.

We will first give the regular definition of Hochschild homology. In Higher Algebra Lurie defined the bimodule ∞-operad $\mathrm{BM}^{\otimes}$ [Lur17, Definition 4.3.1.1]. The ∞-category $\mathrm{BMod}(\mathcal{C})$ of algebras over $\mathrm{BM}^{\otimes}$ in a symmetric monoidal ∞-category $\mathcal{C}$, has as objects, triples (A, B, M), where A, B are algebras and M is an object with a compatible left action of A and a right action of B. There is a forgetful functor, forgetting the object which is acted upon, which gives a cartesian diagram

$$\begin{tikzcd} {}_A\operatorname{BMod}_B(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{BMod}(\mathcal{C}) \arrow[d] \\ A \times B \arrow[r] & \operatorname{Alg}(\mathcal{C}) \times \operatorname{Alg}(\mathcal{C}). \end{tikzcd}$$

We denote the category of R-R bimodules by $\operatorname{BMod}_B(\mathcal{C})$.
Lurie constructed functors taking a left action on an object, to a right action by the opposite algebra, which are equivalences

$$\operatorname{LMod}_{R\otimes R^{\mathrm{op}}} \xleftarrow[\tau_R]{\sim} \operatorname{BMod_R}(\mathcal{C}) \xrightarrow[\sigma_R]{\sim} \operatorname{RMod}_{R\otimes R^{\mathrm{op}}}$$

by [Lur17, Proposition 4.6.3.11]. With this in mind we define Hochschild homology.

Definition 3.3. *Let $\mathcal{C}$ be a symmetric monoidal presentable category, and let $R \in \operatorname{Alg}(\mathcal{C})$ be an associative algebra in $\mathcal{C}$, and let $M \in \operatorname{BMod_R}(\mathcal{C})$ be a bimodule. The Hochschild homology of (R, M) is then*

$$\operatorname{HH}_{\mathcal{C}}(R, M) := \sigma_R(M) \otimes_{R\otimes R^{op}} \tau_R(R) \in \mathcal{C}.$$

Here we use that R is a bimodule over itself by left and right multiplication. The Hochschild homology of R is $\operatorname{HH}(R) := \operatorname{HH}_{\mathcal{C}}(R, R)$.

We want to show that $\operatorname{HH}(R) \simeq \dim(\operatorname{RMod}_R(\mathcal{C}))$. First we show that $\operatorname{RMod}_R(\mathcal{C})$ is dualizable with dual $\operatorname{LMod}_R(\mathcal{C})$. By shifting the multiplication around we have $\operatorname{LMod}_R(\mathcal{C}) \simeq \operatorname{RMod}_{R^{op}}(\mathcal{C})$, so we can construct the duality between these instead.

Lemma 3.4. *The composite*

$$\mathcal{C} \xrightarrow{R^c} \operatorname{RMod}_{R\otimes R^{op}}(\mathcal{C}) \simeq \operatorname{RMod}_R(\mathcal{C}) \otimes_{\mathcal{C}} \operatorname{RMod}_{R^{op}}(\mathcal{C})$$

is the coevaluation of a duality datum, with evaluation

$$\operatorname{RMod}_R(\mathcal{C}) \otimes_{\mathcal{C}} \operatorname{RMod}_{R^{op}}(\mathcal{C}) \simeq \operatorname{RMod}_{R\otimes R^{op}}(\mathcal{C}) \xrightarrow{\otimes_{R\otimes R^{op}} R^e} \mathcal{C}.$$

Here R^c is R with the action shifted to the right, and R^e has the action shifted to the left. Note that these modules also give the above mentioned equivalence

$$\begin{aligned} \operatorname{BMod}_R &\simeq \operatorname{RMod}_{R^{op}\otimes R} \\ M &\mapsto R^c \otimes_{R\otimes R^{op}} (M \boxtimes R^{op}), \end{aligned}$$

where $\boxtimes$ is the external tensor product, sending

$$\operatorname{BMod}_A \times \operatorname{BMod}_B \to \operatorname{BMod}_{A\otimes B}.$$

We are now ready to give the equivalence

Theorem 3.5. *Let $R \in \mathrm{Alg}(\mathcal{C})$ be an associative algebra in $\mathcal{C}$ and let M be an (R,R)-bimodule. There is an equivalence between the Hochschild homology of the pair (R,M) and the $\mathcal{C}$-linear trace of the $\mathcal{C}$-linear endofunctor $-\otimes_R M$: $\mathrm{RMod}_R(\mathcal{C}) \to \mathrm{RMod}_R(\mathcal{C})$:*

$$\mathrm{HH}(R,M) \simeq \mathrm{Tr}_{\mathcal{C}}(\mathrm{RMod}_R(\mathcal{C}), -\otimes_R M) \in \mathcal{C}.$$

In particular for $R = M$, we get

$$\mathrm{HH}(R) \simeq \dim_{\mathcal{C}}(\mathrm{RMod}_R(\mathcal{C})) \in \mathcal{C}.$$

Proof. We will write out the trace explicitly and see that it agrees, with Hochschild homology

$$\mathcal{C} \xrightarrow{\text{coev}} \mathrm{RMod}_R(\mathcal{C}) \otimes_{\mathcal{C}} \ \mathrm{RMod}_{R^{op}} \xrightarrow{M\otimes_{\mathcal{C}} id} \mathrm{RMod}_R(\mathcal{C}) \otimes_{\mathcal{C}} \mathrm{RMod}_{R^{op}} \xrightarrow{\otimes R^e} \mathcal{C}$$

Plugging in our definition of evaluation and coevaluation, we get that it is also given by the composition

$$\mathcal{C} \xrightarrow{R^c} \mathrm{RMod}_{R\otimes R^{\mathrm{op}}} \xrightarrow{\otimes(M\boxtimes R^{\mathrm{op}})} \mathrm{RMod}_{R\otimes R^{\mathrm{op}}} \xrightarrow{\otimes R^e} \mathcal{C}$$

If we evaluate the composition of the first two maps at the unit in $\mathcal{C}$, we get $R^c\otimes(M\boxtimes R^{\mathrm{op}})$, however this is just the image of M in the equivalence $\mathrm{BMod}_R \simeq \mathrm{RMod}_{R^{op}\otimes R}$ given above. If we tensor with R as a left module, then we get exactly the formula for topological Hochschild homology, finishing the proof. □

Example 3.6. *If we choose $\mathcal{C} = \mathrm{Sp}$, then we can take the Hochschild cohomology of an ordinary ring by considering its Eilenberg-Maclane spectrum, which we define to be the topological Hochschild homology $\mathrm{HH}(R)$ of R.*

If we choose $\mathcal{C} = \mathcal{D}(\mathbb{Z})$ with the derived tensor product, then we recover ordinary Hochschild homology.

Lastly we will be interested in the case $\mathcal{C} = \mathcal{D}(\mathrm{Liq}_p)$, where $\mathcal{D}(\mathrm{Liq}_p)$ is the derived ∞-category of complex liquid vector spaces. This is the category Clausen and Scholze defined complex analytic spaces over, and as such will be the setting we will state Hirzebruch-Riemann-Roch in.

Example 3.7. *In the case $\mathcal{C} = \mathrm{Sp}$ we can give a concrete calculation of $\mathrm{HH}(\mathbb{S})$. Since $\mathbb{S}$ is the unit of Sp, we get that $\mathrm{RMod}_{\mathbb{S}}(\mathrm{Sp}) \simeq \mathrm{Sp}$, which is the unit of $\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}}$. However from Example 3.2, this is just equivalent to $\mathbb{S}$, since tensoring with $\mathbb{S}$ is equivalent to the identity.*

3.2 Chern Character from the Universal property of K-theory

We have now established Hochschild homology as the composition

$$\begin{aligned} \mathrm{Alg}(\mathcal{C}) &\xrightarrow{\Theta} (\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}}))^{\mathrm{dbl}} \xrightarrow{\dim_{\mathcal{C}}} \mathcal{C} \\ R &\mapsto \mathrm{RMod}_R(\mathcal{C}) \mapsto \dim_{\mathcal{C}}(\mathrm{RMod}_R(\mathcal{C})), \end{aligned}$$

where Θ is the functor defined in [Lur17, Section 4.8.3] taking an algebra to its category of right modules. We will be interested in a special case of $\mathcal{C}$:

Definition 3.8. *The non-full $(\infty,2)$ sub-category $\mathrm{Cat}^{\mathrm{perf}} \subseteq \mathrm{Cat}_{(\infty,1)}$ is spanned by small, stable and idempotent complete ∞-categories and the exact functors.*

Lurie constructed the fully faithful Ind-completion functor $\mathrm{Ind} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}}$. We define a category $\mathcal{C} \in \mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}}$ to be compactly generated if it is in the image of the Ind-functor. A symmetric monoidal category in $\mathrm{Cat}^{\mathrm{perf}}$, $\mathcal{E} \in \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}})$ is rigid if every object in $\mathcal{E}$ is dualizable.

We now assume that $\mathcal{C} = \mathrm{Ind}(\mathcal{E})$, where $\mathcal{E}$ is rigid. In this case we can recover $\mathcal{E}$ by taking the compact objects in $\mathcal{C}$, $\mathcal{C}^{\omega} \simeq \mathcal{E}$. Recall that $\mathrm{Perf}(R)$ is by definition the compact objects in $\mathrm{RMod}_R(\mathcal{C})$, and that $\mathrm{RMod}_R(\mathcal{C})$ is compactly generated. By [HSS15, Prop 4.9] there is a $\mathcal{C}$-linear ind-completion

$$\mathrm{Mod}_{\mathcal{C}^{\omega}}(\mathrm{Cat}^{\mathrm{perf}}) \to \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})$$

which is fully faithful with image the compactly generated categories. From this we get that Hochschild homology admits the factorization

$$\begin{aligned} \mathrm{Alg}(\mathcal{C}) \to (\mathrm{Mod}_{\mathcal{C}^{\omega}}(\mathrm{Cat}^{\mathrm{perf}}))^{\mathrm{dbl}} &\xrightarrow{\mathrm{Ind}} (\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}}))^{\mathrm{dbl}} \xrightarrow{\dim_{\mathcal{C}}} \mathcal{C}. \\ R \mapsto \mathrm{Perf}(R) &\mapsto \mathrm{RMod}_R(\mathcal{C}) \mapsto \mathrm{Tr}_{\mathcal{C}}(\mathrm{RMod}_R(\mathcal{C})) \end{aligned}$$

We will also write HH for the functor $\dim_{\mathcal{C}} \circ \mathrm{Ind} : \mathrm{Mod}_{\mathcal{C}^{\omega}}(\mathrm{Cat}^{\mathrm{perf}}) \to \mathcal{C}$.
The reason to write Hochschild homology in this way, is that we have another functor defined on perfect complexes of R, namely K-theory.

When $\mathcal{C} = \mathrm{Sp}$, Hochschild homology is the map

$$\mathrm{Cat}^{\mathrm{perf}} \xrightarrow{\mathrm{Ind}} \mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}} \xrightarrow{\dim_{\mathcal{C}}} \mathrm{Sp}\,.$$

We recall from [BGT13] and Hannes' talk, that we have a universal property of K-theory as a localizing invariant $K : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$. Hence, for any localizing invariant E

$$\mathrm{Nat}(K, E) \simeq E(\mathrm{Sp}^{\omega})$$

We will show in the next chapter that Hochschild homology is a localizing invariant, and so we get

$$\mathrm{Nat}(K, \mathrm{HH}) \simeq \mathrm{HH}(\mathrm{Sp}^\omega)$$

However we have that $\mathrm{Sp}^\omega \simeq \mathrm{Mod}_{\mathbb{S}}(\mathrm{Sp}^\omega)$, so $\mathrm{THH}(\mathrm{Sp}^\omega) \simeq \mathrm{THH}(\mathbb{S})$, and we calculated earlier that $\mathrm{THH}(\mathbb{S}) \simeq \mathbb{S}$, so

$$\mathrm{Nat}(K, \mathrm{HH}) \simeq \mathbb{S}.$$

Taking π_0 we have $\pi_0 \, \mathrm{Nat}(K, \mathrm{THH}) \simeq \mathbb{Z}$. The only map monoidal map is the one corresponding to 1, which is the Dennis trace map from K-theory to THH natural in $\mathrm{Cat}^{\mathrm{perf}}$. Again in [HSS15], the Dennis trace map is generalized to other choices of $\mathcal{C}$ than Sp, giving us a map $K \to \mathrm{THH}$.

3.3 The trace is a local invariant

We owe from our last chapter to show that Hochschild homology is an localizing invariant as a functor $\mathrm{Mod}_{\mathcal{C}^\omega}(\mathrm{Cat}^{\mathrm{perf}}) \to \mathcal{C}$. We will therefore prove that trace is a localizing invariant, and from our identification, Hochschild homology is then also a localizing invariant.

For THH to be a localizing invariant, it has to send exact sequences to cofiber sequences.

Definition 3.9. *A sequence*

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

in $\mathrm{Cat}^{\mathrm{perf}}$ *is exact if it is a cofiber sequence and f is fully faithful.*
Similarly a sequence in $\mathrm{Mod}_{\mathcal{C}^\omega}(\mathrm{Cat}^{\mathrm{perf}})$ *is exact, if it is an exact sequence in* $\mathrm{Cat}^{\mathrm{perf}}$ *by the forgetful functor.*

The first step of our definition of HH is taking Ind-completion, so we have to find out what an exact sequence becomes in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})$ after taking Ind-completion. First we note that $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})$ belong to a class of $(\infty, 2)$-categories with nice properties.

Definition 3.10. *An $(\infty, 2)$-category $\mathcal{C}$ is called linear, if the mapping categories are stable and composition induces exact sequences on mapping categories. It is further linearly symmetric monoidal if the map $\mathcal{C}(X, Y) \to \mathcal{C}(X \otimes Z, Y \otimes Z)$ given by tensoring with Z.*

$\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})$ is a linearly symmetric monoidal $(\infty, 2)$-category.

By [HSS15, Prop. 5.4] the ind-completion of an exact sequence in $\mathrm{Cat}^{\mathrm{perf}}$ is a localization sequence in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})^{\mathrm{trl}}$ defined below

Definition 3.11. *A sequence in an linear $(\infty, 2)$-category $\mathcal{C}$*

$$X \xrightarrow{\iota} Y \xrightarrow{\pi} Z$$

is called a localization sequence if the following conditions hold:

- *ι and π have right adjoints ι^r and π^r.*
- *the composite is the 0 object.*
- *the unit $\eta : id_X \to \iota^r\iota$ and the counit for π is an equivalence.*
- *the sequence $\iota\iota^r \to \mathrm{Id}_Y \to \pi^r\pi$ is a cofiber sequence in $\mathcal{C}(Y, Y)$.*

A sequence in $\mathcal{C}^{\mathrm{trl}}$

$$(X, f) \xrightarrow{(\iota,\alpha)} (Y, g) \xrightarrow{(\pi,\beta)} (Z, h)$$

is a localization sequence, if the underlying sequence in $\mathcal{C}$ is a localization sequence, and the maps are right adjointable.

Since exact sequences in $\mathrm{Mod}_{\mathcal{C}^\omega}(\mathrm{Cat}^{\mathrm{perf}})$ correspond to localization sequences in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})^{\mathrm{dbl}} \hookrightarrow \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{l}}_{\mathrm{st}})^{\mathrm{trl}}$, we have to show that the trace sends localization sequences

$$(X, \mathrm{id}_X) \xrightarrow{(\iota,\alpha)} (Y, \mathrm{id}_Y) \xrightarrow{(\pi,\beta)} (Z, \mathrm{id}_Z)$$

to cofiber sequences, for linearly symmetric monoidal $(\infty, 2)$-category $\mathcal{C}$.

Theorem 3.12. *Let $\mathcal{C}$ be a linearly symmetric monoidal $(\infty, 2)$-category, then the trace sends localization sequences on the form*

$$(X, \mathrm{id}_X) \xrightarrow{(\iota,\alpha)} (Y, \mathrm{id}_Y) \xrightarrow{(\pi,\beta)} (Z, \mathrm{id}_Z)$$

to cofiber sequences in $\Omega\mathcal{C}$.

Proof. If we restrict to endomorphisms over a single object $\mathrm{Tr} : \mathcal{C}(X, X) \to \Omega\mathcal{C}$, the functor is exact, since the trace is given by the composite

$$\mathcal{C}(X, X) \xrightarrow{-\otimes \mathrm{id}_{X^\vee}} \mathcal{C}(X \otimes X^\vee, X \otimes X^\vee) \xrightarrow{-\circ\mathrm{coev}} \mathcal{C}(\mathbb{1}, X \otimes X^\vee) \xrightarrow{\mathrm{ev}\circ -} \mathcal{C}(\mathbb{1}, \mathbb{1})$$

so it preserves cofiber sequences, since $\mathcal{C}$ is linearly symmetric. We therefore want to show that

$$(X, \mathrm{id}_X) \to (Y, \mathrm{id}_Y) \to (Z, id_Z)$$

is equivalent to a sequence

$$(Y, a) \to (Y, b) \to (Y, c)$$

where $a \to b \to c$ is a cofiber sequence. For a localization sequence the unit $\eta : id_X \to \iota^r\iota$ and $\epsilon : \pi\pi^r \to id_Z$ are equivalences, so we have a commutative diagram

$$\begin{tikzcd}
(X, \mathrm{id}_X) \arrow[r, "(\iota,\alpha)"] \arrow[d] & (Y, \mathrm{id}_Y) \arrow[r, "(\pi,\beta)"] \arrow[d, equal] & (Z, \mathrm{id}_Z) \\
(Y, \iota\iota^r) \arrow[r, "(\mathrm{id}_Y,\eta)"'] & (Y, \mathrm{id}_Y) \arrow[r, "(id_Y,\epsilon)"'] & (Z, \pi^r\pi) \arrow[u]
\end{tikzcd}$$

Where $\iota\iota^r \to \mathrm{id}_Y \to \pi^r\pi$ is an cofiber sequence, since it is a localization sequence. Note that because of cyclic invariance we have

$$\mathrm{tr}(\iota\iota^r) \simeq \mathrm{tr}(\iota^r\iota) \simeq \mathrm{tr}(id_X)$$
$$\mathrm{tr}(\pi^r\pi) \simeq \mathrm{tr}(\pi\pi^r) \simeq \mathrm{tr}(id_X)$$

So we get that the sequence

$$\mathrm{tr}(X, \mathrm{id}_X) \to \mathrm{tr}(Y, \mathrm{id}_Y) \to \mathrm{tr}(Z, id_Z)$$

is equivalent to a cofiber sequence, and as such the trace preserves cofiber sequences. □

4 Hirzebruch–Riemann–Roch theorem

We will now give an example of the use of Hochschild homology in Algebraic Geometry in a proof of Grothendieck-Riemann-Roch presented by Dustin and Peter Scholze in [CS22]. We will not give the full proof, but show how the map from K-theory to Hochschild homology plays into it.

We first describe the setting of the theorem. Recall that for a complex manifolds with boundary X, We can define its Hodge cohomology.

Definition 4.1. *Let X be a complex manifold, then the Hodge cohomology is given by*

$$\mathrm{Hdg}(X) = R\Gamma(X; \oplus_{i\geq 0}\Omega^i[i] \in \mathcal{D}(\mathrm{Liq}_p)$$

$$\pi_0 \mathrm{Hdg}(X) = \oplus_{i\geq 0} H^i(X; \Omega^i) \in \mathrm{Liq}_p,$$

where Ω^i is the bundle of i-forms. The contravariant functoriality comes from the pullback of differential forms.

We want a Chern character map $\mathrm{ch} : \mathrm{Vect}(X) \to \pi_0 \mathrm{Hdg}(X)$. To get the map, we first construct the first Chern class of line bundles. Recall that we have $\mathrm{Vect}(X) \simeq H^1(X; \mathcal{O}_X^\times)$. The first Chern class is the natural map

$$c_1 : H^1(X; \mathcal{O}_X^\times) \to H^1(X; \Omega^1) \hookrightarrow \pi_0 \mathrm{Hdg}(X)$$

induced by the map $\mathcal{O}_X^\times \to \Omega^1$ given by

$$f \mapsto d\log(f) = \frac{df}{f}$$

The existence of the Chern character then comes from the following theorem

Theorem 4.2. *There exists a unique map $ch : \mathrm{Vect}(X) \to \pi_0 \mathrm{Hdg}(X)$ for a complex manifold X, such that*

1. *$V \mapsto \mathrm{ch}(V)$ commute with pullbacks.*
2. *For a short exact sequence of vector bundles on X*

$$0 \to V' \to V \to V'' \to 0,$$

   we have $\mathrm{ch}(V) = \mathrm{ch}(V') + \mathrm{ch}(V'')$.

3. *for a line bundle $\mathcal{L}$ we have $\mathrm{ch}(\mathcal{L}) = e^{c_1(\mathcal{L})}$.*

$e^{c_1(\mathcal{L})}$ is understood as the Taylor series of e^x. This is a finite sum since the Hodge cohomology vanishes in degrees higher than $\dim(X)$. The Chern character is also multiplicative with the tensor product $\mathrm{ch}(V \otimes W) = \mathrm{ch}(V) \cdot \mathrm{ch}(W)$.

Remark 4.3. *We also have a similar construction of the Todd class, but it is multiplicative with short exact sequences and*

$$\mathrm{Td}(L) = \frac{c_1(L)}{1 - e^{-c_1(L))}}.$$

The Chern character defines a natural map

$$\mathrm{ch} : K_0(\mathrm{Vect}(X)) \to \pi_0 \operatorname{Hdg}(X)$$

Since the Chern character by construction commutes with pullbacks, and it preserves short exact sequences.

Given a proper map $X \to Y$ there is no pushforward structure on $\mathrm{Vect}(-)$, however if we take the larger ∞ category $\mathrm{Perf}(-)$ we have the following proposition.

Lemma 4.4. *Let $f : X \to Y$ be a proper map of compact complex manifolds. Then the pushforward map $f_* : C_X \to C_Y$ sends $\mathrm{Perf}(X)$ to $\mathrm{Perf}(Y)$.*

From this we get a pushforward structure on $K(\mathrm{Perf}(-))$. Hodge cohomology also has pushforward structure, by considering the dual maps of the pullback map by Serre duality.

The question is now, does the Chern character commute with the pushforward structure as well? There is no a priori reason for this to be true, and it is not, but it turns out if we add the Todd class it commutes. In short we have a commutative diagram

$$\begin{tikzcd}
K_0(\mathrm{Perf}(X)) \arrow[r, "f_*"] \arrow[d, "\mathrm{td}(T_x)\cdot \mathrm{ch}"] & K_0(\mathrm{Perf}(Y)) \arrow[d, "\mathrm{td}(T_x)\cdot \mathrm{ch}"] \\
\operatorname{Hdg}(X) \arrow[r, "f_*"'] & \operatorname{Hdg}(Y).
\end{tikzcd}$$

Even the case when $Y = *$ and X is a compact manifold is interesting. Evaluating at $c \in \mathrm{Perf}(X)$, the upper right gives

$$\mathrm{ch}(f_*) \cdot \mathrm{td}(T_*) = \mathrm{ch}(f_*) = \chi(X, c),$$

while the lower left gives

$$f_*(\mathrm{ch}(c) \cdot \mathrm{td}(T_X)) = \int_X \mathrm{ch}(c) \cdot \mathrm{td}(T_X),$$

where the integral is the trace map $\int_X : \mathrm{Hdg}(X) \to \mathbb{C}$ for a compact closed manifold. From this we get the equality

$$\chi(X, c) = \int_X \mathrm{ch}(c) \cdot \mathrm{td}(T_X)$$

which is the statement of Hirzebruch-Riemann-Roch. We now discuss how to use Hochschild-homology to prove Grothendieck-Riemann-Roch theorem.

The proof will use a factorisation of the Chern character

$$K_0(\mathrm{Perf}(X)) \to \pi_0 \mathrm{HH}(X) \cong \pi_0 \mathrm{Hdg}(X)$$

Where the first map is natural with respect to pushforward, and the second is an natural isomorphism, but only with respect to the pullback structure. We will construct the first map.

Recall that we have constructed a natural map $K \to \mathrm{HH}$ as functors from $\mathrm{Mod}_{\mathrm{Perf}(\mathrm{liq_p})}(\mathrm{Cat}^{\mathrm{perf}}) \to \mathrm{D}(\mathrm{liq_p})$. Note a proper map $f : X \to Y$ both induce maps

$$f_* : \mathrm{Perf}(X) \to \mathrm{Perf}(Y)$$
$$f^* : \mathrm{Perf}(Y) \to \mathrm{Perf}(X)$$

The map f_* commutes with colimits and satisfies the projection formula, so it gives a map in $\mathrm{Mod}_{\mathrm{Perf}(\mathrm{liq_p})}(\mathrm{Cat}^{\mathrm{perf}})$, so the map $K \to \mathrm{HH}$ respects the pushforward structure, and so by taking π_0 we get the wanted natural map.

This reduces the proof to comparing the pushforward structure between Hochschild and Hodge cohomology. The rest of the proof can be found in [CS22].

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